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Existence and Stability of a Summable Generalized Solution to a Mathematical Model of a Catalytic Fluidized Bed Process

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Abstract—For a system of first-order partial differential equations describing a catalytic process in a fluidized bed, we consider a mixed problem in the half-strip $0 \le x \le h$, $t \ge 0$. We prove the existence and uniqueness of a bounded summable generalized solution and study its stability. We prove the stabilization as $t \to \infty$ of the values of some physically meaningful functionals of the solution.

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INTRODUCTION

The mathematical models accounting for the influence of the reaction medium on the nonstationary states of the active surface of catalyst are widely used to describe catalytic processes in fluidized beds [1, 2]. In this article, we consider a circulation model of a process in which we can express the motion of the catalyst particles as two interpenetrating flows: the ascending flow the fraction of whose particles is equal to a, $0 < a < 1$; and the descending flow whose fraction is $b = 1 - a$. For a two-stage catalytic reaction of the first order with respect to the intermediate substance, a mathematical model of the process reduces [3] to a mixed problem on the half-strip $\Pi = \{(x,t) : x \in [0,h], t \ge 0\}$:

$$c_x = -pf(c)(1 - au - bv), \qquad c(0,t) = c_0, \tag{1}$$

$$au_t + u_x = q(v - u) - a(1 + f(c))u + af(c), \tag{2}$$

$$bv_t - v_x = q(u - v) - b(1 + f(c))v + bf(c), \tag{3}$$

with the initial and boundary conditions

$$u(\xi,t) = v(\xi,t), \qquad \xi = 0, h, \qquad t > 0,$$

$$u(x,0) = u^0(x), \qquad v(x,0) = v^0(x), \qquad x \in [0,h],$$

$$c_0 \ge 0, \qquad 0 \le u^0(x) \le 1, \qquad 0 \le v^0(x) \le 1,$$

where $a, b, p > 0$, $q \ge 0$, and c_0 are constant parameters; t is time; x is the coordinate along the bed height; $c(x,t)$, $u(x,t)$, and $v(x,t)$ are the concentrations of the substance in the gas phase and in the ascending and descending flows of catalyst particles; and $f(c)$ is a function describing the chemical reaction rate.

In accordance with the current understanding of catalytic processes, we assume that $f(c) = 0$ for $c \le 0$ and $f(c) > 0$ for $c > 0$. Only the solutions satisfying $0 \le c(x,t) \le c_0$, $0 \le u(x,t) \le 1$, and $0 \le v(x,t) \le 1$ are physically meaningful.

A similar mathematical model was used [2] for numerical simulations of catalytic processes. Thus, it is of practical interest to study the qualitative properties of this boundary value problem: the existence of stationary and nonstationary solutions, their stability or instability, and the stabilization of some physically meaningful functionals of the solution.

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This article naturally continues [3–6]. In [3], we proved the existence and uniqueness of a stationary solution to (1)–(3). In [4–6], under various assumptions regarding the functions $f(c)$, $u^0(x)$, and $v^0(x)$, we proved the existence and uniqueness for all $t \geq 0$ of continuously differentiable and continuous generalized solutions to (1)–(3). In [3, 5, 6], for the solution under consideration, we proved the stabilization as $t \to \infty$ of the values of two functionals,

$$W(t) = \Phi(u, v) = \int_0^h (au(x,t) + bv(x,t))\, dx$$

and $c(h,t)$, of the solution with concrete physical meanings to their values on the stationary solution. In [6], we studied the stability of a continuous generalized solution.

In applications the initial conditions $u_0(x)$ and $v_0(x)$ of (1)–(3) can be discontinuous. Thus, the existence, uniqueness, and stability of discontinuous generalized solutions are urgent questions. In this article, we prove the existence and uniqueness of a bounded summable generalized solution to (1)–(3), study its stability, and prove the stabilization of the functionals $c(h,t)$ and $\Phi(u,v)$ as $t \to \infty$.

Henceforth, let $M(Q)$ denote both scalar and vector spaces of functions defined and bounded on a domain Q. Consider the following norms in $M(Q)$:

$$\|w\|_Q = \|w\|_{M(Q)} = \sup_Q |w|, \qquad \|\overline{w}\|_Q = \|\overline{w}\|_{M(Q)} = \max_i \sup_Q |w_i|, \tag{4}$$

where $\overline{w} = (w_1, \dots, w_n)$. Let $C(Q)$ and $H^\alpha(Q)$ denote the spaces of continuous and Hölder continuous functions with the exponent $\alpha \leq 1$. For vectors $U = (u, v)$, we understand under the inequality $U \geq m$ (or $U \leq M$), where m and M are scalars, the simultaneous fulfillment of $u \geq m$ and $v \geq m$ (or $u \leq M$ and $v \leq M$). Let $K, K_1, K_2, \dots$ denote some constants depending only on the initial data of (1)–(3) but independent of t.

1. THE LINEAR PROBLEM

In the half-strip Π, consider the mixed problem for the linear hyperbolic system

$$au_t + u_x = q(v-u) - a(1 + g_1(x,t))u + af_1(x,t), \tag{5}$$

$$bv_t - v_x = q(u-v) - b(1 + g_2(x,t))v + bf_2(x,t) \tag{6}$$

with the initial and boundary conditions

$$u(\xi,t) = v(\xi,t), \qquad \xi = 0, h, \qquad t > 0,$$

$$u(x,0) = u^0(x), \qquad v(x,0) = v^0(x), \qquad x \in [0,h],$$

where $g_i(x,t) \geq 0$ and $f_i(x,t)$ for $i = 1, 2$ are some given bounded functions on Π.

In the space $M(Q)$ of functions defined on a bounded domain Q, choose the set $M^0(Q)$ of all functions $u(x,t)$ summable in Q and integrable over every straight line segment within Q. In the case of vector functions $U(x,t) = (u(x,t), v(x,t))$, where $u(x,t) \in M^0(Q)$ and $v(x,t) \in M^0(Q)$, let $M^0(Q)$ denote also the space equipped with the norm (4). Using a well-known theorem of [7, p. 85], we can show that $M^0(Q)$ are closed in $M(Q)$ and, consequently, are Banach spaces. Say that $U(x,t) \in M^0(\Pi)$ if $U(x,t) \in M^0(\Pi^T)$ for every $T > 0$, where $\Pi^T = \{(x,t) :\ x \in [0,h], 0 \leq t \leq T\}$. Let $M^1(\Pi)$ be the set of vector functions $U(x,t) \in M^0(\Pi)$ such that $u(x,t)$ and $v(x,t)$ are uniformly Lipschitz continuous on Π along the characteristics of (5) and (6) respectively.

Lemma 1. *Take a bounded function $u(x,t)$ on Π that is continuous along the straight lines $a_1x + b_1y = \xi$ and linearly summable along the bounded segments of the lines $a_2x + b_2y = \eta$ within Π, where a_i and b_i are constants satisfying $a_1b_2 \neq a_2b_1$, while ξ and η are some arbitrary parameters. Then*

(a) *$u(x,t) \in M^0(\Pi)$;*

(b) *the integral of $u(x,t)$ along every line segment in Π is a continuous function on Π;*

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(c) *if $u(x,t)$ is uniformly Lipschitz continuous on Π along the lines of the first family then this integral is uniformly Lipschitz continuous on Π.*

Proof. Assume that $u(x,t)$ is summable over x for every $t \geq 0$ and is continuous along the lines $t = ax + \xi$, where $a \neq 0$. For definiteness, assume that $a > 0$. Choosing arbitrary $T > 0$ and integer $n \geq 1$, put $t^k = kT/n$ for $k = 0, 1, 2, \dots, n$ and, in Π^T, define the function

$$u^n(x,t) = \begin{cases} u(0,t^k), & x \leq (t - t^k)/a, \\ u(x - (t-t^k)/a, t^k), & x \geq (t - t^k)/a, \end{cases}$$

$$t^k \leq t < t^{k+1}, \qquad k = 0, 1, \dots, n-1.$$

It is not difficult to verify that $u^n(x,t) \in M^0(\Pi^T)$ for every finite n; moreover,

$$\sup |u^n(x,t)| \leq \sup |u(x,t)|, \qquad \sup |u(x,t) - u^n(x,t)| \to 0 \quad \text{as} \quad n \to \infty.$$

Then the completeness of $M^0(\Pi^T)$ implies $u(x,t) \in M^0(\Pi^T)$. Put

$$I(x,t) = \int_0^x u(\xi,t)d\xi, \qquad (x,t) \in \Pi. \tag{7}$$

For arbitrary $t \geq 0$ and $0 < \delta < ah$, express the difference

$$I(x,t+\delta) - I(x,t)$$
$$= \int_0^{\delta/a} u(\xi, t+\delta)d\xi - \int_{x-\delta/a}^{x} u(\xi,t)d\xi + \int_0^{x-\delta/a} [u(\xi + \delta/a, t+\delta) - u(\xi,t)]d\xi. \tag{8}$$

Since $u(x,t)$ is bounded, (7) yields the uniform Lipschitz continuity of $I(x,t)$ on Π with respect to x. Since $u(x,t)$ are continuous and bounded along the lines parallel to the line $t = ax$; therefore, the right-hand side of (8) vanishes as $\delta \to 0$; consequently, $I(x,t)$ is continuous on Π. If $u(x,t)$ is uniformly Lipschitz continuous on Π along these lines with some Lipschitz constant L then (8) yields

$$|I(x,t+\delta) - I(x,t)| \leq \left(2\|u(x,t)\| + hL\sqrt{1+a^2}\right)\delta/a.$$

Since $I(x,t)$ is uniformly Lipschitz continuous on Π with respect to x, this estimate implies that $I(x,t)$ is uniformly Lipschitz continuous on Π. □

In the general case, the proof goes similarly. The only difference is that to obtain the required estimates we use two equalities similar to (8) with separate increments of t and x.

Let $t_1(x,t) \leq t$ and $t_2(x,t) \leq t$ denote the values of the time variable for which the characteristics of (5) and (6) passing through $(x,t) \in \Pi$ intersect the boundary of Π:

$$t_1(x,t) = \begin{cases} 0, & t \leq ax, \\ t - ax, & t > ax, \end{cases} \qquad t_2(x,t) = \begin{cases} 0, & t \leq b(h-x), \\ t - b(h-x), & t > b(h-x). \end{cases}$$

Integrating (5) and (6) along the characteristics, we obtain the system of integral equations

$$u(x,t) = u^1(x,t) + \int_{t_1}^{t} F_1[u,v](x - (t-\tau)/a, \tau)d\tau, \tag{9}$$

$$v(x,t) = v^1(x,t) + \int_{t_2}^{t} F_2[u,v](x + (t-\tau)/b, \tau)d\tau, \tag{10}$$

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where

$$u^1(x,t) = \begin{cases} u^0(x-t/a), & t_1 = 0, \\ v(0,t_1), & t_1 > 0, \end{cases} \qquad v^1(x,t) = \begin{cases} v^0(x+t/b), & t_2 = 0, \\ u(h,t_2), & t_2 > 0, \end{cases}$$

$$F_1[u,v](x,t) = q_1(v(x,t) - u(x,t)) - (1 + g_1(x,t))u(x,t) + f_1(x,t), \qquad q_1 = q/a,$$

$$F_2[u,v](x,t) = q_2(u(x,t) - v(x,t)) - (1 + g_2(x,t))v(t,x) + f_2(x,t), \qquad q_2 = q/b.$$

Definition 1. Refer to $U(x,t) \in M^0(\Pi)$ satisfying (9) and (10) in Π as a *bounded summable generalized solution* to (5), (6).

Theorem 1. *Take $g_i(x,t) \in M^0(\Pi)$ and $f_i(x,t) \in M^0(\Pi)$ with $g_i(x,t) \geq 0$ for $i = 1,2$, and let $U^0(x) \in M^0[0,h]$. Then the system* (9), (10) *is uniquely solvable in $M^1(\Pi)$, and its solution $U(x,t)$ satisfies*

$$\min\{m_0, m_1, m_2\} \leq U(x,t) \leq \max\{M_0, M_1, M_2\} \tag{11}$$

for all $(x,t) \in \Pi$, where

$$m_0 = \min\{\inf_{0\leq x\leq h} u^0(x), \inf_{0\leq x\leq h} v^0(x)\}, \qquad M_0 = \max\{\sup_{0\leq x\leq h} u^0(x), \sup_{0\leq x\leq h} v^0(x)\},$$

$$m_i = \inf_{\Pi}(f_i(x,t)/(1 + g_i(x,t))), \qquad M_i = \sup_{\Pi}(f_i(x,t)/(1 + g_i(x,t))), \qquad i = 1,2.$$

Proof. We prove the existence of a solution by successive approximations. As the initial approximation, take an arbitrary function $U^1(x,t) \in M^0(\Pi)$. Find the successive approximations $U^n(x,t)$, $n = 2,3,\dots$, by solving

$$u^n(x,t) = u_1^n(x,t) + \int_{t_1}^{t} F_1[u^n, v^{n-1}](x - (t-\tau)/a, \tau)d\tau, \tag{12}$$

$$v^n(x,t) = v_1^n(x,t) + \int_{t_2}^{t} F_2[u^{n-1}, v^n](x + (t-\tau)/b, \tau)d\tau, \tag{13}$$

where

$$u_1^n(x,t) = \begin{cases} u^0(x-t/a), & t \leq ax, \\ v^n(0, t-ax,), & t > a\,x, \end{cases} \qquad v_1^n(x,t) = \begin{cases} v^0(x+t/b), & t \leq b(h-x), \\ u^n(h, t-b(h-x)), & t > b(h-x). \end{cases}$$

Verify that the so-obtained system is solvable in $M^0(\Pi)$ and

$$\|U^n(x,t)\|_\Pi \leq K = \max\left\{\|U^1(x,t)\|_\Pi,\ \|U^0(x)\|_{[0,h]},\ \|f_1(x,t)\|_\Pi,\ \|f_2(x,t)\|_\Pi\right\}$$

for all n. Assuming that $U^{n-1}(x,t) \in M^0(\Pi)$ and $\|U^{n-1}(x,t)\|_\Pi \leq K$, show that

$$U^n(x,t) \in M^0(\Pi), \qquad \|U^n(x,t)\|_\Pi \leq K.$$

The proof goes successively for the domains

$$Q_{1,0} = \{(x,t) :\ 0 \leq x \leq h,\ 0 \leq t \leq ax\},$$
$$Q_{2,0} = \{(x,t) :\ 0 \leq x \leq h,\ 0 \leq t \leq h - bx\},$$

and, for $i = 1,2,\dots$,

$$Q_{1,i} = \{(x,t) :\ 0 \leq x \leq h,\ (i-1)h + ax \leq t \leq ih + ax\},$$
$$Q_{2,i} = \{(x,t) :\ 0 \leq x \leq h,\ ih - bx \leq t \leq (i+1)h - bx\}.$$

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In $Q_{1,0}$. the solution to (12) is of the form

$$u^n(x,t) = u^0(x - t/a)G_1(x,t,0) + \int_0^t G_1(x,t,\tau) f_1^n(x - (t-\tau)/a, \tau)d\tau, \qquad (14)$$

where

$$f_1^n(x,t) = f_1(x,t) + q_1 v^{n-1}(x,t),$$
$$G_1(x,t,\tau) = \exp\left(- \int_\tau^t (1 + q_1 + g_1(x - (t-s)/a, s))ds \right).$$

It follows from (14) that $u^n(x,t) \in M^0(Q_{1,0})$ and $u^n(x,t)$ satisfies $\|u^n(x,t)\|_{Q_{1,0}} \le K$. We can express the solution to (13) in $Q_{2,0}$ as

$$v^n(x,t) = v_0^n(x,t_2)G_2(x,t,t_2) + \int_{t_2}^t G_2(x,t,\tau) f_2^n(x + (t-\tau)/b, \tau)d\tau, \qquad (15)$$

where

$$f_2^n(x,t) = f_2(x,t) + q_2 u^{n-1}(x,t),$$
$$v_0^n(x,t_2) = v_0(x) \text{ for } t_2(x,t) = 0, \qquad v_0^n(x,t_2) = u^n(h,t_2) \text{ for } t_2(x,t) > 0,$$
$$G_2(x,t,\tau) = \exp\left(- \int_\tau^t (1 + q_2 + g_2(x + (t-s)/b, s))ds \right),$$

which implies $v^n(x,t) \in M^0(Q_{2,0})$ and $\|v^n(x,t)\|_{Q_{2,0}} \le K$. Using similar constructions for the remaining domains $Q_{1,i}$ and $Q_{2,i}$, $i = 1, 2, \dots$, we find $U^n(x,t) \in M^0(\Pi)$ and $\|U^n(x,t)\|_\Pi \le K$.

Put $\Delta U^n(x,t) = U^n(x,t) - U^{n-1}(x,t)$ for $n = 2, 3, \dots$. Using (12) and (13), find the equations for $\Delta u^n(x,t)$ and $\Delta v^n(x,t)$ for $n = 2, 3, \dots$:

$$\Delta u^n(x,t) = \Delta u_1^n(x,t) + \int_{t_1}^t \Delta F_1^n(x - (t-\tau)/a, \tau)d\tau, \qquad (16)$$

$$\Delta v^n(x,t) = \Delta v_1^n(x,t) + \int_{t_2}^t \Delta F_2^n(x + (t-\tau)/b, \tau)d\tau, \qquad (17)$$

where

$$\Delta u_1^n(x,t) = \begin{cases} 0, & t_1(x,t) = 0, \\ \Delta v^n(0,t_1), & t_1(x,t) > 0, \end{cases} \qquad \Delta v_1^n(x,t) = \begin{cases} 0, & t_2(x,t) = 0, \\ \Delta u^n(h,t_2), & t_2(x,t) > 0, \end{cases}$$

$$\Delta F_1^n(x,t) = -(1 + q_1 + g_1(x,t))\Delta u^n(x,t) + q_1 \Delta v^{n-1}(x,t), \qquad q_1 = q/a,$$
$$\Delta F_2^n(x,t) = -(1 + q_2 + g_2(x,t))\Delta v^n(t,x) + q_2 \Delta u^{n-1}(x,t), \qquad q_2 = q/b.$$

By (16) in $Q_{1,0}$,

$$\Delta u^n(x,t) = \int_0^t G_1(x,t,\tau) q_1 \Delta v^{n-1}(x - (t-\tau)/a, \tau)d\tau,$$

which implies

$$\|\Delta u^n(x,t)\|_{Q_{1,0}} \le \rho_1 \|\Delta v^{n-1}(x,t)\|_{Q_{1,0}} \le \rho \|\Delta U^{n-1}(x,t)\|_\Pi,$$

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where $\rho_i = q_i/(1 + q_i)$ for $i = 1, 2$, and $\rho = \max\{\rho_1, \rho_2\} < 1$. The solution to (17) in $Q_{2,0}$ satisfies

$$\Delta v^n(x,t) = \Delta v_0^n(x,t_2)G_2(x,t,t_2) + \int_{t_2}^{t} G_2(x,t,\tau)q_2\Delta u^{n-1}(x + (t-\tau)/b, \tau)d\tau;$$

hence, the previous estimate yields $\|\Delta v^n(x,t)\|_{Q_{2,0}} \le \rho\|\Delta U^{n-1}(x,t)\|_\Pi$.

Making similar estimates in $Q_{1,i}$ and $Q_{2,i}$ for $i = 1, 2, \dots$, we obtain

$$\|\Delta U^n(x,t)\|_\Pi \le \rho\|\Delta U^{n-1}(x,t)\|_\Pi, \qquad \rho < 1. \tag{18}$$

Since $M^0(\Pi)$ is complete, this implies the existence of the same limit function $U(x,t)$ for all $U^1(x,t) \in M^0(\Pi)$ satisfying $\|U^n(x,t) - U(x,t)\|_\Pi \to 0$ as $n \to \infty$. Passing to the limit in (12) and (13), we see that $U(x,t)$ is a solution to (9), (10).

In order to verify (11), choose as $U^1(x,t)$ a function satisfying (11). Suppose that U^{n-1} satisfies (11). Using (14), establish that $u^n(x,t)$ satisfies (11) in $Q_{1,0}$. Therefore, we can find from (13) a similar estimate for $v^n(x,t)$ in $Q_{2,0}$. Using similar estimates successively in $Q_{1,i}$ and $Q_{2,i}$ for $i = 1, 2, \dots$, we verify that $U^n(x,t)$ satisfies (11), but then the limit function $U(x,t)$ satisfies this estimate as well. This implies the uniqueness of a solution to (9), (10), while (9), (10), and (11) imply $U(x,t) \in M^1(\Pi)$. The proof of Theorem 1 is complete. □

Corollary 1. *On assuming the hypotheses of Theorem* 1*, suppose that* $f_i(x,t) \ge 0$ *and* $(x,t) \in \Pi$ *for* $i = 1, 2$*, while* $U^0(x) \ge 0$ *and* $x \in [0,h]$*. Then, for all* $(x,t) \in \Pi$*, the solution to the problem* (9)*,* (10) *satisfies* $U(x,t) \ge 0$.

2. THE NONLINEAR PROBLEM

Consider the problem (1)–(3) in the half-strip Π. Let $M_1^0(\Pi)$ denote the space of vector functions $V(x,t) = (c(x,t), u(x,t), v(x,t))$, where $c(x,t) \in C(\Pi)$, $u(x,t) \in M^0(\Pi)$, and $v(x,t) \in M^0(\Pi)$. We can show that $M_1^0(\Pi)$ is closed in $M(\Pi)$ and is a Banach space. Let $M_1^1(\Pi)$ denote the set of vector functions $V(x,t) \in M_1^0(\Pi)$ such that $c(x,t) \in H^1(\Pi)$, while $u(x,t)$ and $v(x,t)$ are uniformly Lipschitz continuous on Π along the characteristics of (2) and (3) respectively. Let S^0 denote the set of vector functions $U^0(x) = (u^0(x), v^0(x)) \in M^0[0,h]$ satisfying $0 \le U^0(x) \le 1$. The set S^0 is closed in $M[0,h]$.

Integrating (1)–(3) over the characteristics, we obtain

$$c(x,t) = c_0 - \int_0^x F_0[V](\xi,t)d\xi, \tag{19}$$

$$u(x,t) = u^1(x,t) + \frac{1}{a}\int_{t_1}^{t} F_1[V](x - (t-\tau)/a, \tau)d\tau, \tag{20}$$

$$v(x,t) = v^1(x,t) + \frac{1}{b}\int_{t_2}^{t} F_2[V](x + (t-\tau)/b, \tau)d\tau, \tag{21}$$

where

$$u^1(x,t) = \begin{cases} u^0(x - t/a), & t_1 = 0, \\ v(0, t - ax), & t_1 > 0, \end{cases} \qquad v^1(x,t) = \begin{cases} v^0(x + t/b), & t_2 = 0, \\ u(h, t - b(h-x)), & t_2 > 0, \end{cases}$$

$$F_0[V](x,t) = pf(c(x,t))(1 - au(x,t) - bv(x,t)),$$

$$F_1[V](x,t) = q(v(x,t) - u(x,t)) - a(1 + f(c(x,t)))u(x,t) + af(c(x,t)),$$

$$F_2[V](x,t) = q(u(x,t) - v(x,t)) - b(1 + f(c(x,t)))v(t,x) + bf(c(x,t)).$$

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It is not difficult to verify that (1)–(3) and (19)–(21) are equivalent in the class of smooth solutions.

Definition 2. Refer to $V(x,t) \in M_1^0(\Pi)$ satisfying (19)–(21) in Π as a *bounded summable generalized solution* to (1)–(3).

Theorem 2. *Suppose that $c_0 \geq 0$, $f(c) \in C[0,c_0]$ for $c_0 > 0$, and $U^0(x) \in S^0$. Then there exists a solution $V(x,t) \in M_1^1(\Pi)$ to (19)–(21) and, for all $(x,t) \in \Pi$,*

$$0 \leq c(x,t) \leq c_0, \qquad 0 \leq u(x,t) \leq 1, \qquad 0 \leq v(x,t) \leq 1. \tag{22}$$

Moreover, if $f(c)$ is Lipschitz continuous then the solution is unique.

Proof. If $c_0 = 0$ then (19) implies $c(x,t) \equiv 0$, and the problem (19)–(21) reduces to (20)–(21) with $f(c) \equiv 0$. In this case, the existence and uniqueness of a solution to (19)–(21), as well as the validity of (22), follow from Theorem 1.

Suppose that $c_0 > 0$. Choosing an arbitrary value of $T \in (0,\infty)$, consider (19)–(21) in the rectangle

$$\Pi^T = \{(x,t) : 0 \leq x \leq h, 0 \leq t \leq T\}.$$

In this case, we can prove the existence of a solution using the Schauder fixed point theorem. To this end, define the operator L on the bounded closed convex set $S = \{z(x,t) \in C(\Pi^T), 0 \leq z(x,t) \leq c^0\}$ as follows: for arbitrary $z(x,t) \in S$, find the functions $u(x,t)$ and $v(x,t)$ as the solution to (20), (21) for $f(c) = f(z)$. Theorem 1 implies that, for every $z(x,t) \in S$, this solution exists, belongs to $M^1(\Pi^T)$, and satisfies $0 \leq U(x,t) \leq 1$. Using $u(x,t)$ and $v(x,t)$, define $c(x,t) = Lz(x,t)$ as the solution to

$$J(c(x,t),c_0) \equiv \int\limits_{c(x,t)}^{c_0} \frac{ds}{f(s)} = \psi^*(x,t) \equiv \begin{cases} \psi(x,t), & \psi(x,t) \leq K, \\ K, & \psi(x,t) \geq K, \end{cases} \tag{23}$$

where $\psi(x,t) = p(x - w(x,t))$, $w(x,t) = \int_0^x (au(\xi,t) + bv(\xi,t))d\xi$, and $K = J(0,c_0)$.

For all $(x,t) \in \Pi^T$, (23) is uniquely solvable [3] for $c(x,t)$. If $\psi(x,t) < K$ then $0 < c(x,t) \leq c_0$, and if $\psi(x,t) \geq K$ then $c(x,t) = 0$. Taking into account (23) for two distinct values of $(x,t) \in \Pi^T$ and $(x_1,t_1) \in \Pi^T$, we find

$$J(c(x,t), c(x_1,t_1)) = \psi^*(x,t) - \psi^*(x_1,t_1) \tag{23'}$$

that implies

$$|c(x,t) - c(x_1,t_1)| \leq \max f(c) |\psi^*(x_1,t_1) - \psi^*(x,t)|.$$

Theorem 1 and Lemma 1 imply $w(x,t) \in H^1(\Pi^T)$, but then $\psi(x,t) \in H^1(\Pi^T)$ and $\psi^*(x,t) \in H^1(\Pi^T)$. Consequently, $c(x,t) \in H^1(\Pi^T)$, and the operator L compactly maps the bounded closed convex set S into its part.

Verify the continuity of L. Take arbitrary functions z_1 and z_2 in S, the solutions u_1, v_1 and u_2, v_2 to (20), (21) with $f(c) = f(z_1)$ and $f(c) = f(z_2)$, and put $c_i = Lz_i$ for $i = 1,2$. Put the pair $\Delta U = U_1 - U_2$ and $\Delta c = c_1 - c_2$. Then $\Delta u(x,t)$, $\Delta v(x,t)$ is a solution to

$$\begin{aligned} \Delta u(x,t) &= \Delta u^1(x,t) + \int\limits_{t_1}^{t} \Delta F_1(x - (t-\tau)/a, \tau)d\tau, \\ \Delta v(x,t) &= \Delta v^1(x,t) + \int\limits_{t_2}^{t} \Delta F_2(x + (t-\tau)/b, \tau)d\tau, \end{aligned} \tag{24}$$

where

$$\Delta u^1(x,t) = \begin{cases} 0, & t_1(x,t) = 0, \\ \Delta v(0,t_1), & t_1(x,t) > 0, \end{cases} \qquad \Delta v^1(x,t) = \begin{cases} 0, & t_2(x,t) = 0, \\ \Delta u(h,t_2), & t_2(x,t) > 0, \end{cases}$$

$$\Delta F_1(x,t) = -(1 + q_1 + f(z_1))\Delta u(t,x) + q_1 \Delta v(x,t) + (f(z_1) - f(z_2))(1 - u_2),$$

$$\Delta F_2(x,t) = -(1 + q_2 + f(z_1))\Delta v(t,x) + q_2 \Delta u(x,t) + (f(z_1) - f(z_2))(1 - v_2).$$

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By Theorem 1, the solution to (24) satisfies

$$\|\Delta U(x,t)\|_{\Pi^T} \leq \|f(z_1) - f(z_2)\|_{\Pi^T}. \tag{25}$$

Using (23′), we obtain

$$|J(c_1(x,t), c_2(x,t))| \leq p\int_0^x (a|\Delta u(\xi,t)| + b|\Delta v(\xi,t)|)d\xi, \tag{26}$$

whence

$$\|\Delta c(x,t)\|_{\Pi^T} \leq ph \max f(c)|\|\Delta U(x,t)\|_{\Pi^T}. \tag{27}$$

The continuity of $f(c)$ together with (25) and (27) imply the continuity of L.

Therefore, all requirements of the Schauder theorem are fulfilled. Consequently, there exists a function $z^* \in S$ with $z^* = Lz^*$. However, then, by the definition of L, the function $c(x,t) = z^*(x,t)$ and the corresponding $u(x,t)$ and $v(x,t)$ satisfy (20), (21) and (23) in Π^T, while $c(x,t) \in H^1(\Pi^T)$, $U(x,t) \in M^1(\Pi^T)$, and (22) holds for all $(x,t) \in \Pi^T$. Since $T > 0$ is arbitrary and (22) is independent of T, it follows that the problem (20), (21), (23) is solvable for all $t \geq 0$.

Verify that the problems (19)–(21) and (20), (21), (23) are equivalent in the class of solutions under consideration. Take a solution $V(x,t)$ to (20), (21), (23) and show that $V(x,t)$ is a solution to (19)–(21). It suffices to show that the function $c(x,t)$ appearing in the solution to (20), (21), (23) satisfies (19). Consider (23) for an arbitrarily chosen value of $t \geq 0$. The function $\psi(x,t)$ in the right-hand side of (23) is continuous and nondecreasing with respect to x, while $\psi(0,t) = 0$. Thus, if $\psi(h,t) \geq K$ then there is a point $x^*(t) \in [0,h]$ with $\psi(x,t) < K$ for $x < x^*(t)$ and $\psi(x,t) \geq K$ for $x \geq x^*(t)$. If $\psi(h,t) < K$ then put $x^*(t) = h$. It follows from (23) that $c(x,t) > 0$ and $f(c(x,t)) > 0$ for all $x < x^*(t)$. For arbitrary values $x \leq x^*(t)$, divide the segment $[0,x]$ into n parts with the points $x_i = ix/n$, $i = 0,1,2,\dots,n$. Since $c(x,t)$ and $f(c)$ are continuous and the integrals $J(c(x_i,t), c(x_{i-1},t))$ are bounded on every segment $[x_{i-1},x_i]$, $i = 1,2,\dots,n$, there is $\xi_i \in [x_{i-1},x_i]$ such that

$$J(c_{i-1},c_i) = \frac{c_{i-1} - c_i}{f(c(\xi_i,t))} = p\int_{x_{i-1}}^{x_i} s(\xi,t)d\xi,$$

where $f(c(\xi_i,t)) > 0$, $s(x,t) = 1 - au(x,t) - bv(x,t)$, and $c_i = c(x_i,t)$ for $i = 1,2,\dots,n$.

Multiplying the resulting equalities by $f(c(\xi_i,t))$ and summing them over i from 1 to n, we obtain

$$c_0 - c(x,t) = p\int_0^x f(c^n(\xi,t))s(\xi,t)d\xi,$$

where $c^n(x,t)$ is a piecewise constant function with respect to x, and $c^n(x,t) = c(\xi_i,t)$ for $x \in [x_{i-1},x_i)$, $i = 1,2,\dots,n$.

Passing in the previous equality to the limit as $n \to \infty$ and considering that

$$\sup_x |f(c(x,t)) - f(c^n(x,t))| \to 0 \quad \text{as} \quad n \to \infty,$$

we arrive at (19). If $x^*(t) < h$ then $c(x,t) = 0$ and $f(c(x,t)) = 0$ for $x \in [x^*(t),h]$; consequently, (19) is valid for all $x \in [0,h]$ and $t \geq 0$, and $c(x,t)$ satisfies (19).

We can prove similarly that every solution to (19)–(21) belonging to $M_1^0(\Pi)$ is a solution to (20), (21), (23). Therefore, in this class, the solutions to (19)–(21) and to (20), (21), (13) are equivalent. Consequently, the problem (19)–(21) is solvable for all $t \geq 0$, and its solution satisfies (22).

Verify the uniqueness of a solution to (19)–(21). Take two distinct solutions $V_1(x,t)$ and $V_2(x,t)$ and put $\Delta V(x,t) = V_1(x,t) - V_2(x,t)$. Then $\Delta U(x,t)$ is a solution to a system similar to (24) with

$$\begin{aligned} \Delta F_1(x,t) &= -(1 + q_1 + f(c_1))\Delta u(t,x) + q_1\Delta v(x,t) + (f(c_1) - f(c_2))(1 - u_2), \\ \Delta F_2(x,t) &= -(1 + q_2 + f(c_1))\Delta v(t,x) + q_2\Delta u(x,t) + (f(c_1) - f(c_2))(1 - v_2). \end{aligned} \tag{28}$$

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Estimating the integrals in the right-hand sides of (24) and (28) for $t \in [0, \tau]$, $\tau > 0$, and taking the Lipschitz continuity of $f(c)$ into account, we obtain

$$\|\Delta U\|_{\Pi^\tau} \le \tau K_1(\|\Delta U\|_{\Pi^\tau} + \|\Delta c\|_{\Pi^\tau}), \tag{29}$$

where $\Pi^\tau = \{(x,t) : 0 \le x \le h, 0 \le t \le \tau\}$.

For $c_1(x,t)$ and $c_2(x,t)$, we have (27). Inserting it in (29), we arrive at

$$\|\Delta U\|_{\Pi^\tau} \le \tau K_2\|\Delta U\|_{\Pi^\tau},$$

where the constants K_1 and K_2 are independent of t and τ. For $\tau < 1/K_2$, the last inequality is valid only for $\|\Delta U\|_{\Pi^\tau} = 0$, which implies the uniqueness of a solution to (19)–(21). □

3. STABILIZATION OF THE VALUES OF FUNCTIONALS OF THE SOLUTION

Theorem 3. *Take $c_0 \ge 0$, $f(c) \in C[0, c_0]$ for $c_0 > 0$, and the solution $V(x,t)$ to (19)–(21) with the initial data $U^0(x) \in S^0$. Then the values of $c(h,t)$ and $\Phi(u,v)$ as $t \to \infty$ monotonely tend to $c^*(h)$ and $\Phi(u^*, v^*)$ on the stationary solution, and, for all $t \ge 0$, we have*

$$|\Phi(u,v) - \Phi(u^*,v^*)| \le |\Phi(u^0,v^0) - \Phi(u^*,v^*)| \exp(-t),$$

$$|c(t,h) - c^*(h)| \le pF|\Phi(u^0,v^0) - \Phi(u^*,v^*)| \exp(-t).$$

Proof. Using (20), for $t \ge 0$ and $0 < \delta < ah$, express the difference

$$a(u(x,t+\delta) - u(x,t)) = a\hat{u}(x,t) + \int_{\tau_1}^{t+\delta} F_1[V](x - (t-\tau)/a, \tau)d\tau, \tag{30}$$

where

$$\hat{u}(x,t) = u(0,\tau_1), \qquad \tau_1 = t + \delta - ax \quad \text{for} \quad 0 \le x < \delta/a,$$

$$\hat{u}(x,t) = 0, \qquad \tau_1 = t \quad \text{for} \quad \delta/a \le x \le h - \delta/a,$$

$$\hat{u}(x,t) = -u(h,\tau_2) + \int_t^{\tau_2} F_1[u,v](x + (\tau - t)/a, \tau)d\tau,$$

$$\tau_1 = t, \qquad \tau_2 = t + a(h-x) \qquad \text{for} \quad h - \delta/a < x \le h.$$

Integrating (30) over x from 0 to h, upon changing the variable and switching the order of integration in the double integrals (which is allowed since $F_1[u,v](x,t) \in M^0(\Pi)$), we arrive at

$$a\int_0^h (u(x,t+\delta) - u(x,t))dx = \int_t^{t+\delta} (u(0,\tau) - u(h,\tau))d\tau + \int_t^{t+\delta}\int_0^h F_1[V](x,\tau)dxd\tau. \tag{31}$$

Similarly using (21), we have

$$b\int_0^h (v(x,t+\delta) - v(x,t))dx = \int_t^{t+\delta} (v(h,\tau) - v(0,\tau))d\tau + \int_t^{t+\delta}\int_0^h F_2[V](x,\tau)dxd\tau. \tag{32}$$

Integrating (19) for $x = h$ over t from t to $t + \delta$ and adding up the left- and right-hand sides of the resulting equalities, (31), and (32), we obtain the integral conservation law for the intermediate substance in the bed:

$$W(t+\delta) - W(t) = -\int_t^{t+\delta} W(\tau)d\tau - p^{-1}\int_t^{t+\delta} (c(h,\tau) - c_0)d\tau, \tag{33}$$

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where $W(t) = \int_0^h (au(x,t) + bv(x,t))dx$ is the total content of the intermediate substance in the bed.

Since $W(t)$ and $c(h,t)$ are continuous; therefore, dividing (33) by δ and passing to the limit as $\delta \to 0$, we arrive at the Cauchy problem

$$W_t(t) = -W(t) - p^{-1}c(h,t) + p^{-1}c_0, \qquad W(0) = \Phi(u^0, v^0). \tag{34}$$

For $c_0 = 0$, we have: $c(h,t) \equiv 0$, the stationary solution to (19)–(21) is $V^*(x) \equiv 0$, and $\Phi(u^*, v^*) = 0$. In this case, (34) yields $\Phi(u,v) = W(t) = \Phi(u^0, v^0)\exp(-t)$.

Suppose that $c_0 > 0$. Generally speaking, the hypotheses of Theorem 3 fail to imply the uniqueness of a solution to (19)–(21), and so, the uniqueness of $W(t)$ either. Using (23), we can show [3] that $c(h,t)$ is a monotonely increasing continuously differentiable function of $W(t)$:

$$c(h,t) = \Psi(W(t)), \qquad \frac{d\Psi(W)}{dW} = pf(\Psi(W)). \tag{35}$$

Consequently, $W(t)$ is uniquely determined for all $t \geq 0$ as the solution to the Cauchy problem (34).

For the stationary solution $V^*(x) = (c^*(x), u^*(x), v^*(x))$ to (1)–(3), and thus, to (19)–(21), the values $c^*(h)$ and $W^* = \Phi(u^*, v^*)$ are uniquely determined [3] and satisfy the stationary equation

$$p^{-1}(c_0 - c^*(h)) = W^* = \Phi(u^*, v^*), \qquad c^*(h) = \Psi(W^*). \tag{36}$$

Using (34) and (36), we arrive at the Cauchy problem

$$\Delta W_t(t) = -\Delta W(t) - p^{-1}\Delta\Psi, \qquad \Delta W(0) = W(0) - W^*,$$

where $\Delta W(t) = W(t) - W^*$ and $\Delta\Psi = c(h,t) - c^*(h) = \Psi(W(t)) - \Psi(W^*)$.

It follows from (35) that $\Delta\Psi \geq 0$ for $\Delta W(t) \geq 0$, and $\Delta\Psi \leq 0$ for $\Delta W(t) \leq 0$. The rest of the proof of Theorem 3 coincides with the proof of Theorem 2 of [3]. $\square$

4. STABILITY OF THE GENERALIZED SOLUTION

Let $H_0^{1/2}[0,h]$ and $H_0^1[0,h]$ denote the spaces of vector functions $V(x) = (c(x), u(x), v(x))$ with the norms

$$\|V(x)\|_{H_0^{1/2}[0,h]} = \max\left\{\|c(x)\|_{H^{1/2}[0,h]},\ \|U(x)\|_{L_2(0,h)}\right\},$$

$$\|V(x)\|_{H_0^1[0,h]} = \max\left\{\|c(x)\|_{H^1[0,h]},\ \|U(x)\|_{M[0,h]}\right\},$$

where

$$\|U(x)\|_{L_2(0,h)} = \left(\int_0^h (au^2(x) + bv^2(x))dx\right)^{1/2}.$$

Given $f(c) \in H^1[0, c_0]$, define $F = \max f(c)$ and $\mu = \min f(c)$ over the segment $c \in [c_h, c_0]$, where $c_h = c_0 \exp(-pF^1 h)$, and F^1 is the Lipschitz constant of $f(c)$.

Theorem 4. *Take $c_0 \geq 0$, and let $f(c) \in H^1[0, c_0]$ with $phFF^1 \leq 2$ for $c_0 > 0$. Then every pair $V_i(x,t) \in M_1^0(\Pi)$, $i = 1, 2$, of generalized solutions to (1)–(3) with initial data $U_i^0(x) \in S^0$ satisfy*

$$\|V_1(x,t) - V_2(x,t)\|_{H_0^{1/2}[0,h]} \leq K_1 e^{-\gamma t}\|U_1^0(x) - U_2^0(x)\|_{L_2([0,h)}, \tag{37}$$

$$\|V_1(x,t) - V_2(x,t)\|_{H_0^1[0,h]} \leq K_2 e^{-\gamma t}\|U_1^0(x) - U_2^0(x)\|_{M[0,h]} \tag{38}$$

for all $t \geq 0$, where $\gamma = 1$ for $c_0 = 0$ and $\gamma = \mu$ for $c_0 > 0$, while K_1 and K_2 are constants independent of t, $U_1^0(x)$, and $U_2^0(x)$.

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Proof. The existence and uniqueness of these generalized solutions follow from Theorem 2. Put $\Delta V(x,t) = V_1(x,t) - V_2(x,t)$. By (20) and (21), we obtain

$$\Delta u(x,t) = \Delta u^1(x,t) + \frac{1}{a}\int\limits_{t_1}^{t} \Delta F_1(x-(t-\tau)/a,\tau)d\tau, \tag{39}$$

$$\Delta v(x,t) = \Delta v^1(x,t) + \frac{1}{b}\int\limits_{t_2}^{t} \Delta F_2(x+(t-\tau)/b,\tau)d\tau, \tag{40}$$

where

$$\Delta F_1(x,t) = q(\Delta v(x,t) - \Delta u(x,t)) - a(1+f(c_1))\Delta u(x,t) + a\Delta f_1(x,t),$$
$$\Delta F_2(x,t) = q(\Delta u(x,t) - \Delta v(x,t)) - b(1+f(c_1))\Delta v(t,x) + b\Delta f_2(x,t),$$
$$\Delta f_1(x,t) = (f(c_1)-f(c_2))(1-u_2), \qquad \Delta f_2(x,t) = (f(c_1)-f(c_2))(1-v_2).$$

Using (39), find an expression for the difference $\Delta u(x,t+\delta) - \Delta u(x,t)$ for $t > 0$ and $t \le \delta \le ah$ similar to (30). Multiply the resulting equality by $\Delta\bar{u}(x,t) = (\Delta u(x,t+\delta) + \Delta u(x,t))/2$ and integrate it over x from 0 to h. Then, using

$$\Delta\bar{u}(x,t) = \Delta\bar{u}(0,t-ax) + \frac{1}{a}\int\limits_{\tau_1}^{t+\delta} \Delta\bar{F}_1(x-(t-\tau)/a,\tau)d\tau, \qquad x < \delta/a,$$

$$\Delta\bar{u}(x,t) = \Delta\bar{u}(h,t-ax) - \frac{1}{a}\int\limits_{t}^{\tau_2} \Delta\bar{F}_1(x-(t-\tau)/a,\tau)d\tau, \qquad x > h-\delta/a,$$

where $\Delta\bar{F}_1(x,t) = (\Delta F_1(x,t+\delta) + \Delta F_1(x,t))/2$, upon changing the variable and switching the order of integration in the double integrals, we arrive at

$$\frac{a}{2}\left(\int\limits_0^h \Delta u^2(x,t+\delta)dx - \int\limits_0^h \Delta u^2(x,t)dx\right) = \int\limits_0^\delta \Delta u(0,t+\tau)\Delta\bar{u}(0,t-\delta+\tau)d\tau$$
$$- \int\limits_0^\delta \Delta u(h,t+\tau)\Delta\bar{u}(h,t+\tau)d\tau + \int\limits_0^\delta\int\limits_0^h \Delta\bar{u}(x,t+\tau)\Delta F_1(x,t+\tau)dxd\tau + O(\delta^2),$$

where $\Delta u^2 = (\Delta u)^2$ and $\Delta v^2 = (\Delta v)^2$. Dividing both sides by δ and passing to the limit as $\delta \to 0$, by the well-known theorems (see [7, pp. 224–225]), we deduce that

$$\frac{a}{2}\left(\int\limits_0^h \Delta u^2(x,t)dx\right)_t = \Delta u^2(0,t) - \Delta u^2(h,t) + \int\limits_0^h \Delta u\Delta F_1(x,t)dx \tag{41}$$

for almost all $t \ge 0$.

Similarly using (40), we find that

$$\frac{b}{2}\left(\int\limits_0^h \Delta v^2(x,t)dx\right)_t = \Delta v^2(h,t) - \Delta v^2(0,t) + \int\limits_0^h \Delta v\Delta F_2(x,t)dx \tag{42}$$

for almost all $t \ge 0$. For $c_0 > 0$ and $U(x,t) \ge 0$, the solution to (19) satisfies

$$c(x,t) \ge c_0\exp(-pF^1x) \ge c_0\exp(-pF^1h) > 0.$$

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Using (41) and (42) with

$$\Delta u^2(0,t) = \Delta v^2(0,t), \qquad \Delta v^2(h,t) = \Delta u^2(h,t), \qquad f(c_1(x,t)) \geq \mu > 0,$$

we arrive at

$$Z_t(t) \leq -2(1+\mu)Z(t) + 2g(t) \tag{43}$$

for almost all $t \geq 0$, where

$$Z(t) = \int_0^h (a\Delta u^2 + b\Delta v^2)dx, \qquad g(t) = \int_0^h |f(c_1) - f(c_2)|\,(a|\Delta u| + b|\Delta v|)dx.$$

By (26),

$$|\Delta c(x,t)| \leq pF\int_0^x (a|\Delta u(\xi,t)| + b|\Delta v(\xi,t)|)d\xi. \tag{44}$$

Consequently,

$$g(t) \leq pFF^1\int_0^h \left(\psi(x,t)\int_0^x \psi(\xi,t)d\xi\right)dx = \frac{1}{2}pFF^1\left(\int_0^h \psi(x,t)dx\right)^2,$$

where $\psi(x,t) = a|\Delta u(x,t)| + b|\Delta v(x,t)|$. Applying the Cauchy–Bunyakovskii inequality to the last estimate while considering that

$$(a|\Delta u| + b|\Delta v|)^2 \leq (a^2+ab)\Delta u^2 + (b^2+ab)\Delta v^2 = a\Delta u^2 + b\Delta v^2$$

for $a \geq 0$, $b \geq 0$, and $a+b=1$, we find that $2g(t) \leq phFF^1Z(t)$. By (43) and the assumption that $hpFF^1 \leq 2$, this yields $Z_t(t) \leq -2\mu Z(t)$ for almost all $t \geq 0$.

Verify that, for all $t > 0$, the absolutely continuous function $Y(t) = Z(0)\exp(-2\mu t) - Z(t)$ is nonnegative. Assume to the contrary that $Y(t^1) < 0$ at some $t^1 > 0$. Since $Y(0) = 0$ and $Y(t)$ is continuous, there exists $t^0 \in [0,t^1)$ such that $Y(t^0) = 0$ and $Y(t) < 0$ for $t \in (t^0,t^1]$. However, then $Y_t(t) \geq -2\mu Y(t) > 0$ for almost all $t \in [t^0,t^1]$, which contradicts the assumption $Y(t^1) < 0$. Consequently, $Y(t) \geq 0$ and, for all $t \geq 0$, we have

$$\|\Delta U(x,t)\|^2_{L^2(0,h)} = Z(t) \leq \|\Delta U^0(x)\|^2_{L^2(0,h)}\exp(-2\mu t). \tag{45}$$

Applying to (44) for $x = h$ successively the Cauchy–Bunyakovskii inequality, the inequality

$$(a|\Delta u| + b|\Delta v|)^2 \leq a\Delta u^2 + b\Delta v^2,$$

and (45), we obtain

$$\|\Delta c(x,t)\|_{C[0,h]} \leq pF\sqrt{h}\sqrt{Z(t)} \leq pF\sqrt{h}\|\Delta U^0(x)\|_{L^2(0,h)}\exp(-\mu t). \tag{46}$$

In order to estimate $\|\Delta U(x,t)\|_{M[0,h]}$, consider

$$y(x,t) = K\exp(-\mu t) \pm \Delta u(x,t), \qquad z(x,t) = K\exp(-\mu t) \pm \Delta v(x,t),$$

where $K = K_1\|\Delta U^0(x)\|_{M[0,h]}$ and $K_1 = 1 + pFF^1h$. Using (39) and (40), we find for $y(x,t)$ and $z(x,t)$ the equations

$$y(x,t) = y_1(x,t) + \frac{1}{a}\int_{t_1}^t G_1[y,z](x-(t-\tau)/a,\tau)d\tau, \tag{47}$$

$$z(x,t) = z_1(x,t) + \frac{1}{b}\int_{t_2}^t G_2[y,z](x+(t-\tau)/b,\tau)d\tau, \tag{48}$$

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where

$$y_1(x,t) = \begin{cases} y(x-t/a,0), & t \le ax, \\ z(0,t-ax), & t > ax, \end{cases} \qquad z_1(x,t) = \begin{cases} z(x+t/b,0), & t \le b(h-x), \\ y(h,t-b(h-x)) & t > b(h-x), \end{cases}$$

$$G_1[y,z](x,t) = q(z(x,t)-y(x,t)) - a(1+f(c_1(x,t)))y(x,t) + ag_1(x,t),$$

$$G_2[y,z](x,t) = q(y(x,t)-z(x,t)) - b(1+f(c_1(x,t)))z(t,x) + bg_2(x,t),$$

$$g_i(x,t) = (1+f(c_1)-\mu)K\exp(-\mu t) \pm \Delta f_1(x,t), \qquad i=1,2.$$

By (46),

$$\|\Delta c(x,t)\|_{C[0,h]} \le pFh\|\Delta U^0(x)\|_{M[0,h]}\exp(-\mu t). \tag{49}$$

Consequently,

$$|\Delta f_i(x,t)| \le pFF^1h\|\Delta U^0(x)\|_{M[0,h]}\exp(-\mu t), \qquad g_i(x,t) \ge 0.$$

Since $y(x,0) \ge 0$ and $z(x,0) \ge 0$; therefore, by Corollary 1, the solutions to (47) and (48) satisfy $y(x,t) \ge 0$ and $z(x,t) \ge 0$ which implies

$$\|\Delta U(x,t)\|_{M[0,h]} \le K_1\|\Delta U^0(x)\|_{M[0,h]}\exp(-\mu t). \tag{50}$$

By (19),

$$|\Delta c(x+\delta,t) - \Delta c(x,t)| \le p\int_x^{x+\delta}(F^1|\Delta c(\xi,t)| + F(a|\Delta u(\xi,t)| + b|\Delta v(\xi,t)|))d\xi.$$

Hence, (45), (46), (49), and (50) yield

$$|\Delta c(x+\delta,t) - \Delta c(x,t)| \le K_1\delta^{1/2}\|\Delta U^0(x)\|_{L_2(0,h)}\exp(-\mu t),$$

$$|\Delta c(x+\delta,t) - \Delta c(x,t)| \le K_2\delta\|\Delta U^0(x)\|_{M[0,h]}\exp(-\mu t).$$

Together with (45), (46), (49), and (50), the last estimates imply (37) and (38) for $c_0 > 0$.

Suppose that $c_0 = 0$. Then $c_1(x,t) \equiv c_2(x,t) \equiv 0$ and $\Delta c(x,t) \equiv 0$. In this case, $\Delta u(x,t)$ and $\Delta v(x,t)$ satisfy (39) and (40) for

$$\Delta F_1 = q(\Delta v - \Delta u) - a\Delta u, \qquad \Delta F_2 = q(\Delta u - \Delta v) - b\Delta v,$$

while, for almost all $t \ge 0$, $Z(t)$ satisfies (43) with $\mu = 0$ and $g(t) \equiv 0$. Arguing as in deriving (45) and (50), we obtain (37) and (38) with $\gamma = 1$ for $U(x,t)$, and consequently for $V(x,t)$. Therefore, the proof of Theorem 4 is complete. □

Corollary 2. *Under the hypotheses of Theorem 4, we have:*

(a) *the generalized solution $V(x,t) \in M_1^0(\Pi)$ to problem (1)–(3) with initial data $U^0(x) \in S^0$ is asymptotically stable in $H_0^1[0,h]$ and $H_0^{1/2}(0,h)$ provided that the perturbed initial data belongs to S^0;*

(b) *as $t \to \infty$, all solutions exponentially converge in the norms of these spaces to a unique stationary solution.*

The existence and uniqueness of a stationary solution to (1)–(3) are proved in [3, 6], and the remaining claims of Corollary 2 follow directly from Theorems 2 and 4.

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